

# **ON THE PROPORTION OF DIGITS IN REDUNDANT NUMERATION SYSTEMS\***

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## 1. INTRODUCTION

In the standard binary numeration system, an  $n$ -bit integer  $N$  is uniquely represented as the sum of powers of 2. Specifically,

$$N = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \dots + a_22^2 + a_12^1 + a_02^0,$$

where  $a_i$  is either 0 or 1. As is common,  $N$  can be represented as an  $n$ -tuple of 0's and 1's where the position of the bit determines the power of 2 involved. For example, in a 4-bit standard binary numeration system,  $N = 0101 = 5$ , since 5 is equivalent to  $2^2 + 2^0$ . Newman [13, p. 2422] suggests that the Chinese used the binary numeration system around 3000 B.C..

Instead of powers of 2's, if Fibonacci numbers are used, then an alternate numeration system (viz. Zeckendorf [14]) occurs in which an integer  $N$  may have more than one representative. That is, let

$$N = a_{n-1}F_{n+1} + a_{n-2}F_n + \dots + a_2F_4 + a_1F_3 + a_0F_2, \quad (1)$$

where  $F_i$  is the  $i$ th Fibonacci number. For example,  $1000 = 0110 = 5$  in the Fibonacci numeration system, where 5 is equivalent to both  $F_5$  and  $F_4 + F_3$ . It is known (e.g. Brown [1]) that an  $n$ -tuple of 0's and 1's is a unique representative of  $N$  if every pair of 1's is separated by at least one 0. Under this restriction, we view 1000 as the representative of 5 and 0110 as the redundant representative. Brown [2] showed that if one represents an integer by the  $n$ -tuple with the *most* 1's, then this representative is unique. In this case, we view 0110 as the representative of 5 and 1000 as the redundant representative.

Representations of this type have important advantages. For example, in a CD-ROM, three or more consecutive 1's cannot be reliably read (Davies [4]). Motivated by this, Klein[11] investigated Fibonacci-like representations of the form (1), where  $F_i = F_{i-1} + F_{i-m}$ , for  $i > m+1$ , and  $F_i = i-1$ , for  $1 < i \leq m+1$ . The case  $m = 2$  corresponds to the Zeckendorf representation using Fibonacci numbers.

Kautz [9] uses such representations in a data transmission system where the receiver clock is synchronized to the transmitter clock using only the data. Toward this end, he uses code words in which there are neither strings of 1's of length greater than  $m$  nor strings of 0's of length greater than  $m$ .

Dimitrov and Donevsky [5] show that the number of steps required to multiply two  $n$ -bit numbers represented in the Zeckendorf numeration system using Quadranacci numbers is less than that required by two numbers represented as standard binary numbers. That is, even though the Zeckendorf representation requires more bits, its efficiency in the multiplication process more than compensates for extra operations because of larger word size. Indeed, the Zeckendorf representation outperforms both standard binary multiplication and multiplication using the most efficient multiplication algorithm for  $n \rightarrow \infty$  when the number of bits in the standard binary representation is 131 through 1200.

The question posed and answered in this paper is: To what extent does redundancy occur in redundant numeration systems? The question has important consequences for both the efficiency of number representations and the transmission of data. We analyze redundancy in two ways 1) the number of distinct representative  $n$ -tuples for some given  $n$ , and 2) the proportion of digits used in non-redundant representatives. Table 1 shows the numeration systems considered and the corresponding recurrences, basis elements, and references.

**TABLE 1 Selected numeration systems, recurrences, and basis elements**

Name	Recurrence	Basis elements	Reference
Standard binary	$F_i = 2 F_{i-1}$	$\dots 2^6 \ 2^5 \ 2^4 \ 2^3 \ 2^2 \ 2^1 \ 2^0$	[7,8,12,13]
Zeckendorf -- Fibonacci	$F_i = F_{i-1} + F_{i-2}$	$\dots 21 \ 13 \ 8 \ 5 \ 3 \ 2 \ 1$	[1,2,3,14]
Gen. Fibonacci -- Tribonacci -- Quadranacci	$F_i = F_{i-1} + F_{i-2} + \dots + F_{i-m}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3} + F_{i-4}$	$\dots 44 \ 24 \ 13 \ 7 \ 4 \ 2 \ 1$ $\dots 56 \ 29 \ 15 \ 8 \ 4 \ 2 \ 1$	[3,9]
Generalization of Fibonacci Numbers	$F_i = F_{i-1} + F_{i-m}$ $F_i = F_{i-1} + F_{i-3}$ $F_i = F_{i-1} + F_{i-4}$	$\dots 13 \ 9 \ 6 \ 4 \ 3 \ 2 \ 1$ $\dots 10 \ 7 \ 5 \ 4 \ 3 \ 2 \ 1$	[11]
$m$ -ary Numbers	$F_i = mF_{i-1} - F_{i-2}$ $F_i = 3F_{i-1} - F_{i-2}$ $F_i = 4F_{i-1} - F_{i-2}$	$\dots 144 \ 55 \ 21 \ 8 \ 3 \ 1$ $\dots 780 \ 209 \ 56 \ 15 \ 4 \ 1$	[11]

## 2. BINARY NUMERATION SYSTEMS

Consider a numeration system in which the basis elements are  $(\dots, F_4, F_3, F_2)$ , where

$$F_i = F_{i-1} + F_{i-2} + \dots + F_{i-m},$$

for  $i > m+1$ , and  $F_i = 2^{i-2}$ , for  $1 < i \leq m+1$ , where  $m \geq 2$ . Consider a representative  $n$ -tuple  $T = (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$ , where  $a_i \in \{0,1\}$ . From [3, 6], if no more than  $m-1$  consecutive  $a_i$ 's are 1, then  $T$  is a unique representative of

$$N = \sum_{i=0}^{n-1} a_i F_{i+2}.$$

We can write the regular expression [10, pp. 617-623] for the allowed representatives as

$$\mathbf{R} = (\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \dots + \mathbf{1}^{m-1})(\mathbf{0}(\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \dots + \mathbf{1}^{m-1}))^* . \quad (2)$$

Here,  $\mathbf{a}^* = \{\lambda, a, aa, aaa, \dots\}$ , where  $\lambda$  is the *empty* string, and  $\mathbf{1}^i$  denotes  $i$  consecutive 1's. Thus, this expression represents the set of strings consisting of substrings beginning with  $i$  1's, for  $0 \leq i \leq m-1$ , followed by a sequence of substrings each of the form 0, 01, 011,  $\dots$ , and  $01^{m-1}$ .

From (2), we can derive a generating function  $N(x,y,z)$  for the number of representatives and the number of 0's and 1's in these representatives. Let

$x$  track the number of bits,  
 $y$  track the number of 0's, and  
 $z$  track the number of 1's.

Then, a typical term in the power series expansion of  $N(x,y,z)$  is  $\xi_{nij}x^ny^iz^j$ , for  $n = i + j$ , where  $\xi_{nij}$  is the number of representative  $n$ -tuples with  $i$  0's and  $j$  1's. We can write

$$N(x, y, z) = \left(1 + xz + x^2z^2 + \dots + x^{m-1}z^{m-1}\right) \left( \frac{1}{1 - xy(1 + xz + x^2z^2 + \dots + x^{m-1}z^{m-1})} \right), \quad (3)$$

where the first term represents the leftmost substring, which can be nothing, 1,  $1^2$ , ..., or  $1^{m-1}$ , while the second term represents the ways to choose 0, 01,  $01^2$ , ..., and  $01^{m-1}$ . We can rewrite (3) as follows

$$N(x, y, z) = \left( \frac{1 - x^m z^m}{1 - xz} \right) \left( \frac{1}{1 - xy \left( \frac{1 - x^m z^m}{1 - xz} \right)} \right).$$

From this, we can generate, for example, the distribution of 16-tuples with  $i$  1's, for  $0 \leq i \leq 15$ , as shown in Fig. 1. It is interesting that the number of representative  $n$ -tuples increases markedly from  $m = 2$  to  $m = 3$ , and, for  $m = 7$ , the distribution is almost binomial. The fact that it is not exactly binomial can be seen by its asymmetry. Capocelli, Cerbone, Cull, and Hollaway [3] derive an expression for the average proportion,  $P_{1's}$ , of bits that are 1, when the number  $n$  of bits is large. Table 2 shows this. In the Zeckendorf numeration system using Fibonacci numbers

**TABLE 2 [3,11] Average proportion of 1's in numeration systems with basis elements  $F_i = F_{i-1} + F_{i-2} + \dots + F_{i-m}$ , when the number of bits is large**

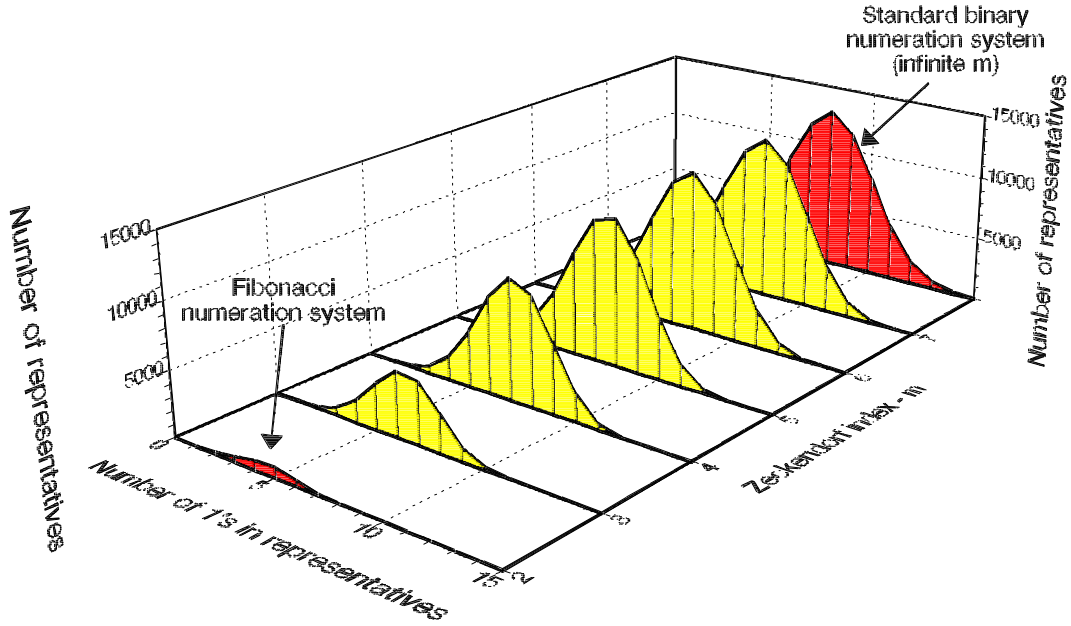
$m$	2	3	4	5	6	7	8	$\infty$
$P_{1's}$	<b>0.2764</b>	<b>0.3816</b>	<b>0.4337</b>	<b>0.4621</b>	<b>0.4782</b>	<b>0.4875</b>	<b>0.4929</b>	<b>0.5000</b>

( $m = 2$ ), the average proportion of 1's is near 25%. However, as  $m$  increases from 2, this value approaches 50%.

Klein [11] considers numeration systems based on the recurrence

$$F_i = F_{i-1} + F_{i-m},$$

for  $i > m + 1$ , and  $F_i = i - 1$ , for  $1 < i \leq m + 1$ , where  $m \geq 2$ . Consider a representative  $n$ -tuple  $T = (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$ , where  $a_i \in \{0,1\}$ . From [6, Th. 1], it follows that, if every pair of 1's



**FIGURE 1. Distributions of 1's in 16-tuple Zeckendorf numeration systems**

is separated by at least  $m-1$  0's, then  $T$  is a unique representative of

$$N = \sum_{i=0}^{n-1} a_i F_{i+2}.$$

For  $m = 2$ , this is the Fibonacci numeration system in which no two 1's are adjacent. A regular expression for the allowed representatives is

$$\mathbf{R} = \mathbf{0}^* + (\mathbf{0} + \mathbf{10}^{m-1})^* \mathbf{10}^*. \quad (4)$$

The  $\mathbf{1}$  in  $\mathbf{10}^*$  represents the rightmost 1 in a string containing at least one 1. In this case, any number of 0's, as described by  $\mathbf{0}^*$  occurs to its right.  $(\mathbf{0} + \mathbf{10}^{m-1})^*$  represents a string consisting of a sequence of substrings of the form 0 and  $\mathbf{10}^{m-1}$ . It follows from this construction that each pair of 1's is separated by at least  $m-1$  0's.

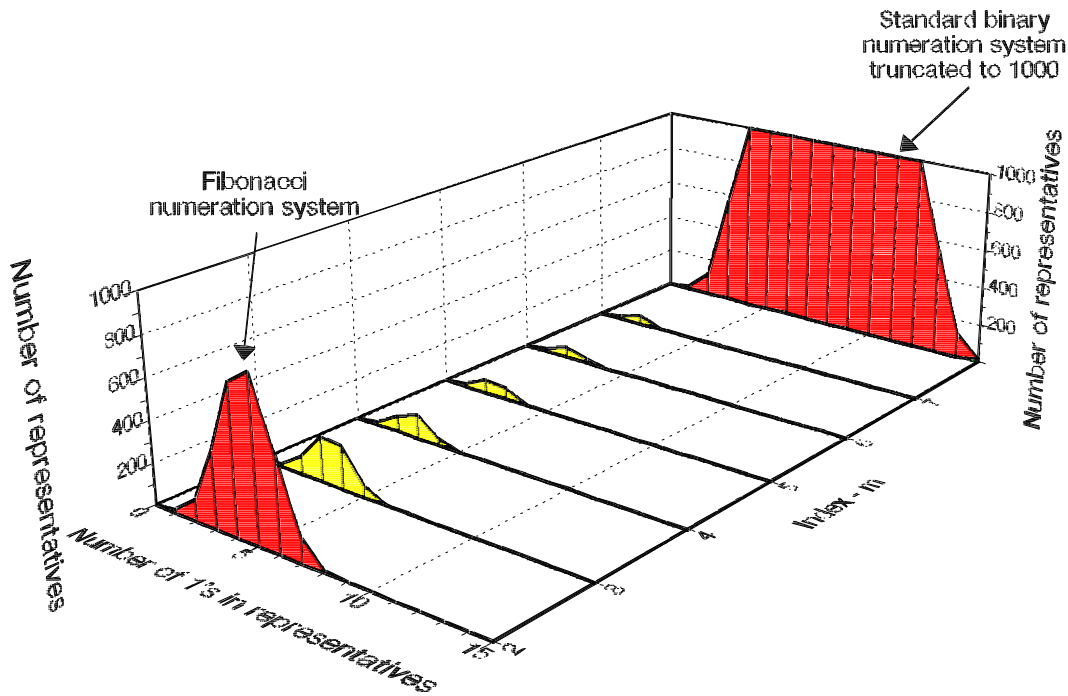
Consider a generating function  $N(x,y,z)$  to count the representative  $n$ -tuples and the 0's and 1's in these representatives. From (4), we can write

$$N(x,y,z) = (1+xy+x^2y^2+\dots) + [(1+(xy+x^m y^{m-1}z)+(xy+x^m y^{m-1}z)^2+\dots) xz (1+xy+x^2y^2+\dots)]. \quad (5)$$

Here,  $(1+xy+x^2y^2+\dots)$  counts the ways to choose no 0's, one 0, two 0's etc., while  $(xy+x^my^{m-1}z)$  counts the ways to choose either a single 0 or  $10^{m-1}$ , and  $xz$  counts the choice of a single 1. Equivalent to (5) is the following

$$N(x, y, z) = \frac{1}{1-xy} \left[ 1 + \frac{xz}{1-xy-x^my^{m-1}z} \right]. \quad (6)$$

From this, we obtain the distribution of 16-tuples according to the number of 1's, as shown in Fig. 2. It is interesting that, even for small  $m$ , the number of representative  $n$ -tuples is small compared to the standard binary numeration system, shown here truncated to 1000 in order to display detail.



**FIGURE 2. Distributions of 1's in 16-tuple numeration systems whose basis elements are generated by the recurrence  $F_i = F_{i-1} + F_{i-m}$**

If we substitute 1 for  $y$  and  $z$  in (6), we achieve a generating function for the number of representative  $n$ -tuples, as follows

$$N(x, 1, 1) = \frac{1}{1-x} \left[ 1 + \frac{x}{1-x-x^m} \right] = \frac{1-x^m}{(1-x)(1-x-x^m)}. \quad (7)$$

Specifically,  $\xi_{n\Box\Box}$ , the coefficient of  $x_n$  in the power series representation of (7), is the number of representative  $n$ -tuples. We can write (7) as

$$N(x,1,1) = \frac{g_m}{1 - \frac{x}{\alpha_m}} + \dots, \quad (8)$$

where ... represents terms whose contribution to  $\xi_{n\Box\Box}$  is negligible, for large  $n$ , compared to the term shown, and  $g_m = \frac{1}{(1-\alpha_m)(1+m\alpha_m^{m-1})}$ .  $\alpha_m$  is the dominant root; i.e. the singularity on the circle of convergence of  $N(x,1,1)$ . We are interested in the value of  $\xi_{n\Box\Box}$  when  $n$  is large, and so we write

$$\xi_{n\Box\Box} \sim \frac{1}{(1-\alpha_m)(1+m\alpha_m^{m-1})} \left( \frac{1}{\alpha_m} \right)^n, \quad (9)$$

where  $f_n \sim g_n$  means  $\lim_{n \rightarrow \infty} \frac{f_n}{g_n} = 1$ .

Consider now the proportion of bits that are 0 and 1 in the representatives counted by  $\xi_{n\Box\Box}$ . Substituting 1 for  $z$  in (6), yields  $N(x,y,1)$ , a generating function in which a typical term is  $(\xi_{n0\Box} + \xi_{n1\Box} y^1 + \xi_{n2\Box} y^2 + \dots + \xi_{nm\Box} y^n) x^n$ , where  $\xi_{ni\Box}$  is the number of representative  $n$ -tuples with  $i$  0's. Differentiating  $N(x,y,1)$ , with respect to  $y$  and setting  $y = 1$  yields a generating function in  $x$  in which a typical term is  $(\xi_{n1\Box} + 2\xi_{n2\Box} + \dots + n\xi_{nm\Box}) x^n = \Xi_n x^n$ . Dividing  $\Xi_n$  by  $\xi_{n\Box\Box}$  yields the average number of 0's in representative  $n$ -tuples. Dividing this by  $n$  gives the average proportion  $P_{0's}$  of bits that are 0. That is,

$$N_{0's}(x) = \sum_{n \geq 0} \Xi_n x^n = \frac{d}{dy} N(x,y,1) \Big|_{y=1} = \frac{(1-x^m)(x + (m-1)x^m)}{(1-x)(1-x-x^m)^2} + \dots, \quad (10)$$

where ... represents terms whose contribution to  $\Xi_n$  is negligible, for large  $n$ , compared to the term shown. But,  $N_{0's}(x)$  can be expressed as

$$N_{0's}(x) = \frac{g_m^2 (1-\alpha_m)(1+(m-1)\alpha_m^{m-1})}{\left(1 - \frac{x}{\alpha_m}\right)^2} + \dots, \quad (11)$$

where ... represents negligible terms.. Therefore, from (11), we have

$$\Xi_n \sim g_m^2 (1-\alpha_m)(1+(m-1)\alpha_m^{m-1}) \left( \frac{1}{\alpha_m} \right)^n n.$$

Thus, the proportion of digits that are 0 when the number  $n$  of digits is large is

$$P_{0's} = \frac{1+(m-1)\alpha_m^{m-1}}{1+m\alpha_m^{m-1}}.$$

Substituting 1 for  $y$  in (6), yields  $N(x,1,z)$ , a generating function in which a typical term is  $(\xi_{n0} + \xi_{n1} z^1 + \xi_{n2} z^2 + \dots + \xi_{nn} z^n) x^n$ , where  $\xi_{nj}$  is the number of representative  $n$ -tuples with  $j$  1's. Differentiating  $N(x,1,z)$  with respect to  $z$  and setting  $z = 1$  yields a generating function in  $x$  in which a typical term is  $(\xi_{n1} + 2\xi_{n2} + \dots + n\xi_{nn}) x^n = \Xi_n x^n$ . Dividing  $\Xi_n$  by  $\xi_{n\Box\Box}$  yields the average



number of 1's in representative  $n$ -tuples. Dividing this by  $n$  gives the average proportion of digits that are 1. That is,

$$N_{1's}(x) = \sum_{n \geq 0} \Xi_n x^n = \frac{d}{dy} N(x, 1, z) \Big|_{z=1} = \frac{(1-x^m)}{(1-x-x^m)^2} + \dots \quad (12)$$

By a similar calculation, we can write for the proportion of bits that are 1

$$P_{1's} = \frac{1 - \alpha_m}{\alpha_m (1 + m\alpha_m^{m-1})} \quad .$$

Table 3 summarizes these results and shows values for the proportion of 0's and 1's for large  $n$ . As  $m$  grows, the proportion of bits that are 1 approaches 0.0 .

**TABLE 3 Asymptotic approximations to the number of representative  $n$ -tuples and the proportion of 0's and 1's in numeration systems with basis elements  $F_i = F_{i-1} + F_{i-m}$**

$m$	Number of representative $n$ -tuples	Proportion of bits that are 0	Proportion of bits that are 1	$\alpha_m$
<b>General <math>m</math></b>	$\frac{1}{(1-\alpha_m)(1+m\alpha_m^{m-1})} \left(\frac{1}{\alpha_m}\right)^n$	$\frac{(1+(m-1)\alpha_m^{m-1})}{(1+m\alpha_m^{m-1})}$	$\frac{(1-\alpha_m)}{\alpha_m(1+m\alpha_m^{m-1})}$	Dominant root of $1-x-x^m$
<b>2</b>	1.1708×1.6180 <sup>n</sup>	0.7236	0.2764	0.6180
<b>3</b>	1.3134×1.4656 <sup>n</sup>	0.8057	0.1943	0.6823
<b>4</b>	1.4397×1.3803 <sup>n</sup>	0.8492	0.1508	0.7245
<b>5</b>	1.5550×1.3247 <sup>n</sup>	0.8762	0.1238	0.7549
<b>6</b>	1.6621×1.2852 <sup>n</sup>	0.8948	0.1052	0.7781
<b>7</b>	1.7630×1.2554 <sup>n</sup>	0.9084	0.0916	0.7965
<b>8</b>	1.8587×1.2320 <sup>n</sup>	0.9188	0.0812	0.8117

### 3. MULTIPLE-VALUED NUMERATION SYSTEMS

There has been less work on numeration systems with non-binary digits. Klein [11] considers numeration systems based on the recurrence

$$F_i = m F_{i-1} - F_{i-2} ,$$

for  $i > 3$ ,  $F_3 = m$ , and  $F_2 = 1$ , where  $m \geq 3$ . Consider a representative  $n$ -tuple  $T = (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$ , where  $a_i \in \{0, 1, \dots, m-1\}$ . From [11], if every pair of  $m-1$ 's is separated by at least one  $i$ , such that  $i \in \{0, 1, \dots, m-3\}$ , then  $T$  is a unique representative of

$$N = \sum_{i=0}^{n-1} a_i F_{i+2} .$$

For this numeration system, we seek the proportion of digits that are 0, 1, ...,  $m-2$ , and  $m-1$ . We use a generating function  $N(x,y,z,w)$  in which

$x$  tracks the number of digits,  
 $y$  tracks the number of  $m-1$ 's,  
 $z$  tracks the number of  $m-2$ 's, and  
 $w$  tracks the number of  $i$ 's,

where  $i$  is a digit restricted by  $0 \leq i \leq m-3$ . By symmetry, the proportion of digits that are  $i$  is the same for any value of  $i$ . Because it is convenient, we choose  $i = 0$ . We enumerate a representative according to whether it has 1) no  $m-1$ 's or 2) at least one  $m-1$ . For 1), there is no restriction on the digits, and the representatives are described by the regular expression

$$\mathbf{P} = (\mathbf{0} + \mathbf{1} + \mathbf{2} + \dots + \mathbf{m-2})^*.$$

The power series expression for the number of representatives, in this case, is

$$1 + (wx + (m-3)x + zx) + (wx + (m-3)x + zx)^2 + (wx + (m-3)x + zx)^3 + \dots \quad (13)$$

That is, the term  $wx$  represents a choice of a 0, which contributes 1 to the count of 0's, as tracked by  $w$ , and 1 to the count of digits, as tracked by  $x$ . Similarly, the term  $zx$  tracks the number of  $m-2$ 's. The term  $(m-3)x$  tracks the number of digits in  $\{1, 2, \dots, m-3\}$ . (13) can be written as

$$\frac{1}{1 - wx - (m-3)x - zx}. \quad (14)$$

For 2), the regular expression that describes the allowed representatives is

$$(\mathbf{P} + (\mathbf{m-1}) (\mathbf{m-2})^* (\mathbf{0} + \mathbf{1} + \mathbf{2} + \dots + (\mathbf{m-3})))^* (\mathbf{m-1}) \mathbf{P}.$$

Here the rightmost  $\mathbf{m-1}$  is the rightmost  $m-1$  in the string. To its right is any substring consisting of the digits 0, 1, ..., and  $m-2$ , as described by  $\mathbf{P}$  and enumerated by (14). The digits to the left of the rightmost  $m-1$  can be chosen from 0, 1, 2, ...,  $m-2$  and from strings beginning in  $m-1$ , ending in a digit whose value is  $m-3$  or less with no, one, two, etc.  $m-2$ 's in between. The choices for the digits to the left of the rightmost  $m-1$  are enumerated by

$$1 + \left[ wx + (m-3)x + zx + \frac{yx[wx + (m-3)x]}{1 - zx} \right] + \left[ wx + (m-3)x + zx + \frac{yx[wx + (m-3)x]}{1 - zx} \right]^2 + \dots$$

Here, the choices of a substring beginning in  $m-1$  are enumerated by  $yx[wz + (m-3)x]/(1-zx)$ , where  $yz$  represents the choice of the first digit  $m-1$ ,  $[wz + (m-3)x]$  represents choice of the last digit, 0, 1, ...,  $m-2$ , and  $1/(1 - zx)$  represents the choice of the  $m-2$ 's in between. Thus, the generating function for the choices of representatives is

$$N(x, y, z, w) = \frac{1}{1 - wx - (m-3)x - zx} \left[ 1 + \frac{yx}{1 - wx - (m-3)x - zx - \frac{yx[wx + (m-3)x]}{1 - zx}} \right]. \quad (15)$$

Substituting 1 for  $y$ ,  $z$  and  $w$  into (15) yields  $N(x, 1, 1, 1)$ , where

$$N(x, 1, 1, 1) = \frac{1}{1 - mx + x^2} \quad (16)$$

is the generating function for the number of representative  $n$ -tuples in this numeration system. Specifically,  $\xi_{n\Box\Box\Box}$ , the coefficient of  $x_n$  in the power series representation of (16), is the number of representative  $n$ -tuples. We prefer to write (16) as

$$N(x) = \frac{g_m}{1 - \frac{x}{\alpha_m}} + \frac{h_m}{1 - \frac{x}{\beta_m}}, \quad (17)$$

where  $\alpha_m = \frac{m - \sqrt{m^2 - 4}}{2}$ ,  $\beta_m = \frac{m + \sqrt{m^2 - 4}}{2} \left( = \frac{1}{\alpha_m} \right)$ ,  $g_m = \frac{1}{1 - \alpha_m^2}$ , and  $h_m = \frac{1}{1 - \beta_m^2}$ . That

is, from (17), we can write  $\xi_{n\Box\Box\Box} = g_m (1/\alpha_m)^n + h_m (1/\beta_m)^n$ . We are interested in the value of  $\xi_{n\Box\Box\Box}$  when  $n$  is large, and so we write

$$\xi_{n\Box\Box\Box} \sim \frac{1}{1 - \alpha_m^2} \left( \frac{1}{\alpha_m} \right)^n. \quad (18)$$

Table 4 shows the values of  $g_m$  and  $1/\alpha_m$  for various  $m$ .

Substituting 1 for  $y$  and  $z$  in (15) yields  $N(x, 1, 1, w)$ . A typical term in the power series representation of this generating function is  $(\xi_{n\Box\Box\Box 0} + \xi_{n\Box\Box\Box 1} w^1 + \xi_{n\Box\Box\Box 2} w^2 + \dots + \xi_{n\Box\Box\Box n} w^n) x^n$ , where  $\xi_{n\Box\Box\Box k}$  is the number of representative  $n$ -tuples with  $k$  0's. Differentiating this with respect to  $w$  and setting  $w = 1$  yields a generating function in  $x$  in which a typical term is  $(\xi_{n\Box\Box\Box 1} + 2 \xi_{n\Box\Box\Box 2} + \dots + n \xi_{n\Box\Box\Box n}) x^n = \Xi_n x^n$ . Dividing  $\Xi_n$  by  $\xi_{n\Box\Box\Box}$  yields the average number of 0's in representative  $n$ -tuples. Dividing this by  $n$  gives the average proportion of digits that are 0. That is,

$$N_{0's}(x) = \sum_{n \geq 0} \Xi_n x^n = \frac{d}{dw} N(x, 1, 1, w) \Big|_{w=1} = \frac{x}{(1 - mx + x^2)^2}. \quad (19)$$

But,  $N_{0's}(x)$  can be expressed as

$$N_{0's}(x) = \frac{\frac{\alpha_m}{(1 - \alpha_m^2)^2}}{\left(1 - \frac{x}{\alpha_m}\right)^2} + \dots, \quad (20)$$

where ... represents terms whose contribution to  $\Xi_n$  is negligible, for large  $n$ , compared to the contributions from the term shown. Therefore, from (20), we have

$$\Xi_n \sim \frac{\alpha_m}{(1-\alpha_m^2)^2} \left( \frac{1}{\alpha_m} \right)^n n.$$

Thus, the proportion of digits that are 0 when the number of digits is large is

$$P_{0's} = \frac{\alpha_m}{1-\alpha_m^2}.$$

By an earlier observation, we can write  $P_{m-3's} = \dots = P_{1's} = P_{0's}$ . Similarly, for the  $m-2$ 's, we have

$$N_{m-2's}(x) = \sum_{n \geq 0} \Xi_n x^n = \frac{d}{dz} N(x, 1, z, 1) \Big|_{z=1} = \frac{(1-x^2)x}{(x^2 - mx + 1)^2} = \frac{\frac{\alpha_m}{1-\alpha_m^2}}{\left(1 - \frac{x}{\alpha_m}\right)^2} + \dots, \quad (21)$$

where ... represents terms that can be neglected, when  $n$  is large. Therefore, from (21), we have

$$\Xi_n \sim \frac{\alpha_m}{1-\alpha_m^2} \left( \frac{1}{\alpha_m} \right)^n n,$$

and

$$P_{m-2's} = \alpha_m.$$

Similarly, for the  $m-1$ 's, we have

$$N_{m-1's}(x) = \sum_{n \geq 0} \Xi_n x^n = \frac{d}{dy} N(x, y, 1, 1) \Big|_{y=1} = \frac{(1-x)^2 x}{(x^2 - mx + 1)^2} = \frac{\frac{\alpha_m(1-\alpha_m)}{1+\alpha_m}}{\left(1 - \frac{x}{\alpha_m}\right)^2} + \dots, \quad (22)$$

where ... represents terms that can be neglected, when  $n$  is large. Therefore, from (22), we have

$$\Xi_n \sim \frac{\alpha_m}{(1+\alpha_m)^2} \left( \frac{1}{\alpha_m} \right)^n n,$$

and

$$P_{m-1's} = \frac{\alpha_m(1-\alpha_m)}{1+\alpha_m}.$$

Table 4 shows the various proportions. Note that, as  $m$  grows, the proportion of digits that are  $i$  for  $0 \leq i \leq m-1$  becomes nearly equal.

**TABLE 4 Asymptotic approximations to the number of representative  $n$ -tuples and proportion of digits in numeration systems with basis elements  $F_i = mF_{i-1} - F_{i-2}$**

$m$	Number of representative $n$ -tuples	Proportion of digits that are $i$ for $0 \leq i \leq m-3$	Proportion of digits that are $m-2$	Proportion of digits that are $m-1$	$\alpha_m$
<b>General <math>m</math></b>	$\frac{1}{1-\alpha_m^2} \left( \frac{1}{\alpha_m} \right)^n$	$\frac{\alpha_m}{1-\alpha_m^2}$	$\alpha_m$	$\frac{\alpha_m(1-\alpha_m)}{(1+\alpha_m)}$	$\frac{m-\sqrt{m^2-4}}{2}$
<b>3</b>	$1.1708 \times 2.6180^n$	0.4472	0.3820	0.1708	0.3820
<b>4</b>	$1.0774 \times 3.7321^n$	0.2887	0.2679	0.1547	0.2679
<b>5</b>	$1.0455 \times 4.7913^n$	0.2182	0.2087	0.1366	0.2087
<b>6</b>	$1.0303 \times 5.8284^n$	0.1768	0.1716	0.1213	0.1716
<b>7</b>	$1.0217 \times 6.8541^n$	0.1491	0.1459	0.1087	0.1459
<b>8</b>	$1.0164 \times 7.8730^n$	0.1291	0.1270	0.0984	0.1270
$\infty$	$1.0000 \times m^n$	$1/m$	$1/m$	$1/m$	$1/m$

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